Propositional mathematical logic can be loosely described as the study of truth and falsehood in the context of statements relating the concepts of a theory. In this section we will take a formal look at the propositional logic and the predicate logic from a relatively intuitive perspective.

1 Propositional Logic

The following are the basic notions of propositional logic.

1. A *sentence* is a *proposition*, or a sequence of *words* in a *language*.
2. A sentence is *well-formed* if it obeys the rules of the language and has exactly one *interpretation*.

A *logical connective* is a symbol used to denote a basic logical operation on sentences. The following is a table of the logical connectives used in the common propositional logic.

Name	**Symbol**	**Syntax**	**English**
Negation	$\neg$	$\neg P$	not
Disjunction	$\vee$	$P \vee Q$	either-or
Conjunction	$\wedge$	$P \wedge Q$	both-and, but
Implication	$\to$	$P \to Q$	if-then, implies, when, if, only if
Biconditional	$\leftrightarrow$	$P \leftrightarrow Q$	if and only if, precisely when

Remark 1. It is important to note that certain of the english connectives for implication are "backwards implications" and others are "forwards implications." In particular, let P and Q be sentences. The following are all synonymous:

$$\begin{array}{lll} P \to Q & \text{if } P \text{ then } Q & P \text{ implies } Q \\ Q \text{ when } P & Q \text{ if } P & P \text{ only if } Q \end{array}$$

Thus there may be multiple different looking English versions of a symbollic statement...

Now it is time to formalize the propositional logic; we must declare both the allowed symbols in sentences and the allowed rules for constructing new sentences from old. Propositional logic uses an *alphabet* (the allowed symbols) as follows:

Name:	**Allowed Symbols**
Sentance Symbols:	Uppercase English Letters
Sentence Variables:	Lowercase English Letters
Logical Connectives:	$\neg$, $\vee$, $\wedge$, $\to$, $\leftrightarrow$
Grouping Symbols:	(,), [,], {, }

Propositional logic has *syntax* (the allowed constructions) as follows: A sentence must be a string of characters in the alphabet obtained by finitely many applications of the following two rules:

1. Every sentence symbol and every sentence variable is a sentence.
2. If $\mathbb{A}$ and $\mathbb{B}$ are sentences, then all of the following are sentences:

$$(\neg\mathbb{A}) \qquad (\mathbb{A} \vee \mathbb{B}) \qquad (\mathbb{A} \wedge \mathbb{B}) \qquad (\mathbb{A} \to \mathbb{B}) \qquad (\mathbb{A} \leftrightarrow \mathbb{B})$$

It is important to notice that when building sentences, we must carefully include the parentheses at each step (lest we end up with something like $A \vee B \wedge C$ again!). However, we might omit the outermost parentheses and use different grouping symbols to improve readability.

Example 1. All of the sentences from previous examples are indeed sentences. None of the following are sentences:

$$A\neg \qquad \neg \wedge A \qquad \vee p \wedge Q \qquad I\ Love\ Maths$$

2 Predicate Logic

Predicate logic is a powerful generalization of propositional logic. Essentially, predicate logic allows us to break up sentences into smaller components. We do so with some new concepts.

1. An *object* is something that appears in a statement.
2. A *predicate* is a sentence describing properties of objects.
3. A *quantifier* is a symbol that denotes the *scope* of a variable object in a statement; a quantifier specifies "how many" objects are involved.

The objects and predicates are intuitive; for instance, 2 is an object. Moreover, "2 is even" is a predicate applied to the object 2 whereas "x is even" is a predicate applied to the variable object x. This is clearly a refinement of the notion of proposition from above. Quantifiers come in two flavors. An *existential quantifier* asserts the existence of an object to satisfy a predicate, while a *universal quantifier* asserts that all objects satisfy a predicate.

Name	**Symbol**	**Syntax**	**English**
Existential Quantifier	$\exists$	$\exists x[P(x)]$	there is, there exists, some
Universal Quantifier	$\forall$	$\forall x[P(x)]$	for all, every

Sometimes it will be convenient to write certain statements in predicate logic; I will try to keep this to a minimum.

3 Translation

It is important to be able to translate between English statements and symbollic statements.

Example 2. Let the following be given traslations:

$$A: \text{ Alex graduates from college}$$
$$B: \text{ Bailey hires Alex}$$
$$C: \text{ Alex becomes a telemarketer}$$

The following are English sentences along with their propositional translations.

1. Bailey hires Alex or Alex becomes a telemarketer. $B \vee C$
2. Alex does not become a telemarketer. $\neg C$
3. Alex becomes a telemarketer if and only if Bailey hires Alex. $C \leftrightarrow B$
4. Both Alex graduates from college and Bailey hires Alex. $A \wedge B$
5. Bailey hires Alex if Alex graduates from college. $A \to B$
6. If Bailey hires Alex, then Alex graduates from college. $B \to A$
7. If Alex does not graduate from college, then Alex becomes a telemarketer. $(\neg A) \to C$

The following are propositions along with English translations.

1. $A \vee (\neg B)$ Either Alex graduates from college or Bailey does not hire Alex.
2. $A \wedge (\neg C)$ Alex graduates from college and does not become a telemarketer.
3. $B \leftrightarrow (\neg C)$ Bailey hires Alex precisely when Alex does not become a telemarketer.
4. $A \wedge (B \wedge C)$ Alex graduates from college and both Bailey hires Alex and Alex becomes a telemarketer.
5. $(A \wedge B) \wedge C$ Both both Alex graduates from college and Bailey hires Alex and Alex becomes a telemarketer.
6. $A \to (B \wedge C)$ If Alex graduates from college, then both Bailey hires Alex and Alex becomes a telemarketer.

Notice that the translations in the previous example are completely unambiguous. In particular, there is exactly one way to translate from each statement to the next. This is good; in mathematics we strive to present results rigorously (i.e. in well-reasoned ways without any ambiguity).

Remark 2. Let A and B be statements. The statement "A and B or C" is ambiguous; indeed, it could have totally different meanings depending on where we put parentheses! We must make our translations unambiguous.

4 Truth Values and Truth Tables

The most natural question to ask about a sentence is whether or not it is true. Indeed, this is the most interesting question to ask in the context of our propositional logic system.

The possible answers to the above question are *True* or *False*. These are called the *truth values* for our system.

Now recall that every sentence is built up from smaller sentences using connectives. Thus we naturally would like a way of understanding truth values of big sentences in terms of the truth values of their components through the connectives. We will do this via *truth table*, listing in a compact chart all possible truth values.

4.1 Negation

The negation changes the truth value of a statement to its opposite.

p	$\neg p$
T	F
F	T

4.2 Disjunction

The disjunction $p \vee q$ is true exactly when either p is true or q is true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Remark 3. Sometimes in English we use "p or q" to mean that not both p and q are true. That "or" is called *exclusive-or*, and is not what we mean in maths by the word "or." We always use the *inclusive-or*, as given above.

4.3 Conjunction

The conjunction $p \wedge q$ is true exactly when both p is true and q is true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

4.4 Implication

The implication $p \to q$ is true exactly when p is true means that q is also true. Thus if p is false, we do not care what q is because we didn't say anything about that case; in such cases $p \to q$ is true. On the other hand if p is true, then q must also be true for $p \to q$ to be true.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Remark 4. The above might seem weird. Think about the following statements:

$$P: \text{ I ate eggs for breakfast}$$
$$Q: \text{ I am full during my maths class}$$

Clearly $P \to Q$; supposing I did not eat eggs for breakfast (i.e. P is false), I might be hungry during maths (maybe I didn't eat at all) or I might not be hungry during maths (perhaps I ate a bagel). In either case, the implication holds because I only said what happens if P is true. Supposing I did eat eggs for breakfast (i.e. P is true) but I was hungry during maths class, then $P \to Q$ would be false because the hypothesis did not imply the conclusion!

4.5 Biconditional

The biconditional $p \leftrightarrow q$ is true exactly when both p and q have the same truth value.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

4.6 More Complicated Statements

To determine the truth value of a statement, one applies the following procedure:

1. Write out all components of the statement.
2. Determine the truth value (by cases) of each component of the statement, building up to the big statement.
3. Record these in a truth table.

Example 3. We compute the truth tables of the statements $p \wedge (\neg p)$ and $p \to [(\neg p) \to q]$ and $p \vee (q \to r)$.

p	$\neg p$	$p \wedge (\neg p)$
T	F	F
F	T	F

p	q	$\neg p$	$(\neg p) \to q$	$p \to [(\neg p) \to q]$
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

p	q	r	$q \to r$	$p \vee (q \to r)$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T
F	F	F	T	T

Thus there are three types of truth tables. Some have all false in the final column, some have all true in the final column, and some have a mixture of true and false. These are called *contradictions*, *tautologies*, and *contingencies* respectively.

Two statements $\mathbb{A}$ and B are *logically equivalent* (denoted $\mathbb{A} \equiv \mathbb{B}$) when they have the same truth tables.

Example 4 (Material Implication)**.** The statements $(p \to q) \equiv [(\neg p) \vee q]$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg p$	$(\neg p) \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Hence examining the final columns of these truth tables we have $(p \to q) \equiv [(\neg p) \vee q]$ as desired.

Example 5 (DeMorgan's Laws)**.** We have both $[\neg(p \vee q)] \equiv [(\neg p) \wedge (\neg q)]$ and $[\neg(p \wedge q)] \equiv [(\neg p) \vee (\neg q)]$. Indeed

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

p	q	$\neg p$	$\neg q$	$(\neg p) \vee (\neg q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

p	q	$p \vee q$	$\neg(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

p	q	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

5 Arguments

Truth tables are nice, algorithmically computable objects. However, they can be tedious to compute. Moreover, they are not particularly natural; we don't think in terms of truth tables on a day-to-day basis. Rather, we deduce new knowledge from prior knowledge via certain accepted rules of inference.

5.1 Useful Equivalences

Here are some useful logical equivalences which hold for all sentences p, q, and r. We also include the set theoretic interpretations, where A, B, and C are sets in a universal set U.

Name	Logical Equivalence	Set Interpretation
Double Negation	$\neg(\neg p) \equiv p$	$U \setminus (U \setminus A) = A$
DeMorgan Laws	$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$ $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$	$U \setminus (A \cup B) = (U \setminus A) \cap (U \setminus B)$ $U \setminus (A \cap B) = (U \setminus A) \cup (U \setminus B)$
Material Implication	$p \to q \equiv (\neg p) \vee q$	$A \subseteq B \leftrightarrow U = (U \setminus A) \cup B$
Contraposition	$p \to q \equiv (\neg q) \to (\neg p)$	$A \subseteq B \leftrightarrow U \setminus B \subseteq U \setminus A$
Biconditional Expansion	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$	$A = B \leftrightarrow [(A \subseteq B \text{ and } B \subseteq A)]$
Commutativity	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Associativity	$p \vee (q \vee r) \equiv (p \vee q) \vee r$ $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
Distributivity	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$ $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$
Tautology	$p \vee (\neg p) \equiv \mathbf{T}$	$A \cup (U \setminus A) = U$
Contradiction	$p \wedge (\neg p) \equiv \mathbf{F}$	$A \cap (U \setminus A) = \emptyset$
Simplification	$p \vee \mathbf{T} \equiv \mathbf{T}$ and $p \vee \mathbf{F} \equiv p$ $p \wedge \mathbf{T} \equiv p$ and $p \wedge \mathbf{F} \equiv \mathbf{F}$	$A \cup U = U$ and $A \cup \emptyset = A$ $A \cap U = A$ and $A \cap \emptyset = \emptyset$

Problem 1. Verify that the above are logical equivalences via truth table.

We can use these equivalences to prove additional equivalences!

Example 6. We show that $[\neg(p \vee q)] \to (\neg q)$ is a tautology using the rules of inference. Indeed:

$$\begin{aligned}
[\neg(p \vee q)] \to (\neg q) &\equiv [\neg(\neg q)] \to [\neg(\neg[p \vee q])] && \text{Contraposition} \\
&\equiv q \to (p \vee q) && \text{Double Negation} \\
&\equiv (\neg q) \vee (p \vee q) && \text{Material Implication} \\
&\equiv (p \vee q) \vee (\neg q) && \text{Commutativity} \\
&\equiv p \vee (q \vee (\neg q)) && \text{Associativity} \\
&\equiv p \vee \mathbf{T} && \text{Tautology} \\
&\equiv \mathbf{T} && \text{Simplification}
\end{aligned}$$

5.2 Rules of Deduction

The above rules have limitations; they only allow us to prove equivalences of statements. They cannot be used (for example) to prove general implications. For that we need rules of deduction.

A *rule of deduction* is a tautology of the form $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$. An *argument* is a series of *premises* (i.e. accepted statements) and *deduced statements*, each of which follows from the previous statements by a rule of deduction.[1] The final statement of an argument is the *conclusion*. Oftentimes we will write arguments (and rules of

[1] Each p_i is a premise of the the rule of deduction

deduction) in the form $p_1, p_2, \cdots, p_n \therefore q$. The following are the formal ways of stating the most common rules of deduction, together with some nice set theoretic interpretations thereof.

Name	**Rule**	**Interpretation**
Modus Ponens	$p \to q, p \quad \therefore q$	If $A \subseteq B$ and $x \in A$, then $x \in B$.
Modus Tollens	$p \to q, \neg q \quad \therefore \neg p$	If $A \subseteq B$ and $x \notin B$, then $x \notin A$.
Disjunctive Syllogism	$p \vee q, \neg p \quad \therefore q$ $p \vee q, \neg q \quad \therefore p$	The set $(A \cup B) \setminus A \subseteq B$. The set $(A \cup B) \setminus B \subseteq A$.
Hypothetical Syllogism	$p \to q, q \to r \quad \therefore p \to r$	If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
Dilemma	$p \vee q, p \to r, q \to r \quad \therefore r$	If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.
Reductio Ad Absurdum	$(\neg p) \to [q \wedge (\neg q)], \quad \therefore p$	If $x \notin A$ implies $x \in \emptyset$, then $x \in A$.
Appeal to Equivalence	$p \leftrightarrow q, p \quad \therefore q$ $p \leftrightarrow q, q \quad \therefore p$	If $A = B$ and $x \in A$, then $x \in B$. If $A = B$ and $x \in B$, then $x \in A$.
Equivalence Syllogism	$p \leftrightarrow q, q \leftrightarrow r \quad \therefore p \leftrightarrow r$	If $A = B$ and $B = C$, then $A = C$.
Conjunctive Simplification	$p \wedge q \quad \therefore p$ $p \wedge q \quad \therefore q$	The set $A \cap B \subseteq A$. The set $A \cap B \subseteq B$
Conjunctive Addition	$p, q \quad \therefore p \wedge q$ $p, q \quad \therefore q \wedge p$	If $x \in A$ and $x \in B$, then $x \in A \cap B$. If $x \in A$ and $x \in B$, then $x \in B \cap A$.
Disjunctive Addition	$p \quad \therefore p \vee q$ $q \quad \therefore p \vee q$	The set $A \subseteq A \cup B$. The set $B \subseteq A \cup B$.

Problem 2. Verify that the rules of deduction above are valid by truth table.

Problem 3. Convince yourself that the rules of deduction above make sense in the context of English arguments.

5.3 Understanding Arguments and Spotting Fallacies

We will soon see that the rules of deduction above are the foundation for various methods of proof. Indeed, to decide whether or not a conclusion follows from an argument, we need only verify the premises of the argument and subsequently verify that each step of the argument follows from some rule of deduction. Thus Modus Ponens will convince us that the statement we deduced is indeed true.

Problem 4. Convince yourself that Modus Ponens makes sense...

We have seen that to establish the truth of a proposition, we must make an argument using true premises and using the rules of deduction. Often folks will make arguments that they think establish some sort of claim. How do we decide whether or not these arguments are any good? One route is to merely show that both the premises are true and that the argument follows from rules of deduction (thus confirming the claim).

But what if we think that their argument is bad? There are two ways to contest an argument.

1. Question the truth of a premise.
2. Show that some step of the argument was not given via a rule of deduction.

Questioning a premise can be difficult and often relies on the world views of the individuals making the arguments. However, in theory we know how to do the second one. Write down the argument in propositional logic, and then compute a truth table for the rule of deduction that the arguer used at each step! An argument is a *fallacy* when it has a false row in its truth table; i.e. an argument is a fallacy when it is not derived from a tautologous statement.

Remark 5. In general, one can be very clever about how one evaluates arguments. This is (unfortunately) far away from what we will do here as the situation can get very complicated very quickly.

The following are some common fallacies that folks will appeal to when making their arguments.

Name	**Fallacy**
Converse Error	$p \to q, q \quad \therefore p$
Inverse Error	$p \to q, \neg p \quad \therefore \neg q$
Non Sequitur	$p \quad \therefore q$
Circular Argument	$p \to q, q \to p \quad \therefore p$
Affirming a Disjunct	$p \vee q, p \quad \therefore \neg q$
Bad Analogy	$p \vee q, p \quad \therefore q$
Insufficient Evidence	$(p \wedge q) \to r, p \quad \therefore r$
Post Hoc	$p, q \quad \therefore p \to q$
Generalized Particular	$\exists x[P(x)] \quad \therefore \forall x[P(x)]$

Problem 5. Show that each of the above arguments is a fallacy.

6 Set Theory

Another way of looking at logic is through *set theory.* A *set* is a *collection* of *objects* which are *elements* of the set.

For a set S, we write $s \in S$ to say that s is an element of S. Alternatively, we can use a logical statement to denote a set in *set-builder notation.* The most basic form of this notation looks like $\{x : P(x)\}$ where P is a *predicate* describing a property of the elements x in our set. Here are some common sets to illustrate the use of this notation:

$$\begin{aligned}
\mathbb{N} &= \{n : n \text{ is a natural number}\} \\
\mathbb{Z} &= \{n : n \text{ is an integer}\} \\
\mathbb{Q} &= \{r : r \text{ is a rational number}\} \\
\mathbb{J} &= \{x : x \text{ is an irrational number}\} \\
\mathbb{R} &= \{x : x \text{ is a real number}\}
\end{aligned}$$

If a set is small enough, we can merely list out all of its elements between braces. For example $\{1, 2, 3\}$ is a set.

A very important set is the *empty set.* This is the set $\emptyset = \{\}$ which does not have any elements. Another important set is the *universal set*; this changes in various contexts.

This language is very closely related to the language we have previously discussed. In fact, all of the logical connectives have an interpretation in the set theoretic language. The first few are called *operations* on sets, because they take old sets and use them to create new sets. The basic set operations are given in the table below. Let A and B be sets that live in some universal set X.

Name	**Notation**	**Definition**	**Logic Analogue**
Difference	$A \setminus B$	$\{x : x \in A \text{ but } x \notin B\}$	Negation
Union	$A \cup B$	$\{x : x \in A \text{ or } x \in B\}$	Disjunction
Intersection	$A \cap B$	$\{x : x \in A \text{ and } x \in B\}$	Conjunction

The interpretation of implication is the *subset relation.* We say A is a *subset* of B when every element of A is also an element of B; we denote this by $A \subseteq B$. The subset relation is just the predicate statement

$$\forall x[(x \in A) \to (x \in B)]$$

The interpretation of biconditional is the *set equality relation.* We say A is *equal* to B when an object belongs to A exactly when it belongs to B; we denote this by $A = B$. The equality relation is just the predicate statement

$$\forall x[(x \in A) \leftrightarrow (x \in B)]$$

The interpretation of T is the universal set, and the interpretation of F is the emptyset; this is because $x \in U$ is always a true statement and $x \in \emptyset$ is always a false statement.

Example 7. The following are all true relations:

1. $\mathbb{R} \setminus \mathbb{Q} = \mathbb{J}$
2. $\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}$
3. $\mathbb{Q} \cup \mathbb{J} = \mathbb{R}$
4. $\mathbb{Q} \cap \mathbb{J} = \emptyset$
5. $\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$